Interlace Polynomials for Delta-Matroids

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Abstract

We show that the (single variable) interlace polynomial and several related graph polynomials may be defined for set systems (or more specifically, delta-matroids) in general. In this way, and using combinatorial properties of set systems and delta-matroids rather than graph theoretical arguments, we find that various known results about these polynomials, including their recursive relations, are both more efficiently and more generally obtained. In addition, we obtain several interrelationships and results for polynomials on set systems (and delta-matroids) that correspond to new interrelationships and results for the corresponding polynomials on graphs. Finally, we show that the Tutte polynomial for matroids on the diagonal is a special case of the generalized interlace polynomial for delta-matroids, and we obtain in this way novel evaluations of the Tutte polynomial. In particular we prove a conjecture by Las Vergnas.

Keywords: interlace polynomial, delta-matroid, Tutte-Martin polynomial, principal pivot transform, local complementation

1. Introduction

The Martin polynomial [17], which is defined for both directed and undirected graphs, computes the number of circuit partitions of that graph. More precisely, the number a_k of k-component circuit partitions in a graph D is equal to the coefficient of y^k of the Martin polynomial of D. In case D is a 2-in, 2-out digraph, an Euler circuit C of D defines a circle graph G. The interlace polynomial of G turns out to coincide (up to a trivial transformation) with the Martin polynomial of (the underlying 2-in, 2-out digraph) D — the interlace polynomial is however defined for graphs in general (i.e., not only for circle graphs) [2, 3]. The interlace polynomial is known to be invariant under the graph operations of local and edge complementation (where local complementation is defined here only for looped vertices) and moreover the interlace polynomial fulfills a recursive reduction relation. As demonstrated in [1, 4], the interlace polynomial of

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a graph G may be explicitly defined through the nullity values of the set of subgraphs of G. The interlace polynomial moreover coincides with the Tutte-Martin polynomial defined for isotropic systems [3, 9] in the case where graph G does not have loops. Restricting to such simple graphs is an essential loss of generality as, e.g., the Tutte-Martin polynomial does not explain the invariance of the interlace polynomial under local complementation.

Related polynomials have been defined such as the different "interlace polynomial" $Q(G)$ of [1], which is invariant under local, loop, and edge complementation. In case G is a circle graph (with possibly loops), then $Q(G)$ coincides with the Martin polynomial of a 4-regular graph corresponding to G. Moreover, $Q(G)$ coincides with the "global" Tutte-Martin polynomial defined for isotropic systems [9] in the case where graph G is simple. As $Q(G)$ is invariant under loop complementation, this is not an essential loss of generality. Also, the bracket polynomial for graphs [20] has recursive relations similar to the recursive relation of the interlace polynomial.

The notion of delta-matroid (or Δ-matroid), introduced in [6], is a generalization of the notion of matroid. In addition, Δ-matroids may be viewed as a generalization of graphs. Moreover, as pointed out in [15], the graph operations of local and edge complementation have particularly simple interpretations in terms of Δ-matroids. In [12] a suitable generalization of the notion of loop complementation for graphs has been defined for Δ-matroids. Moreover, in [11], the notion of nullity for graph is shown to correspond to a natural distance measure within Δ-matroids.

Using these generalizations of the notions of nullity, and of local, loop, and edge complementation, we generalize in this paper interlace polynomials (and related graph polynomials) to Δ-matroids (Section 4). We find that the invariance properties and recursive relations hold for Δ-matroids in general (Section 6). In fact, using combinatorial properties of Δ-matroids (and its operations), it turns out that the proofs of these results are not only more general but also much more concise compared to the proofs which rely on graph theoretical arguments. Moreover, we obtain various interrelationships and evaluations for polynomials on Δ-matroids that correspond to new interrelationships and evaluations for the corresponding polynomials on graphs (Sections 7 and 10 for Δ-matroids and graphs, respectively).

Inspired by the connection between the Martin and the Tutte polynomial in [17] (and its relation to isotropic systems described in [7]) we find that the Tutte polynomial $t_M(x, y)$ for matroids M can, for the case $x = y$, be seen as a special case of the generalized interlace polynomial for Δ-matroids. In addition, the recursive relations of the generalized interlace polynomial and $t_M(y, y)$ coincide when restricting to matroids M. The obtained evaluations of the generalized interlace polynomial are then straightforwardly carried over to $t_M(y, y)$. In particular we prove in this way a conjecture by Las Vergnas [16, Conjecture 4.2].

2. Distance in Set Systems and Δ-matroids

In this section we recall the algebra of set systems generated by the operations of pivot and loop complementation [12]. Additionally we recall from [11] the notion of distance in set systems and its main properties w.r.t. Δ-matroids.

We let $\mathbb{F}_2$ be the field consisting of two elements. In this field addition and multiplication are equal to the logical exclusive-or and logical conjunction, which are denoted by $\oplus$ and $\wedge$ respectively. Of course these operations carry over to sets, e.g., for sets $A, B \subseteq V$ and $x \in V$, $x \in A \oplus B$ iff $(x \in A) \oplus (x \in B)$. For sets, $\oplus$ is thus the symmetric difference operator.

A *set system* (over V) is a tuple $M = (V, D)$ with V a finite set called the *ground set* and $D \subseteq 2^V$ a family of subsets of V. Let $X \subseteq V$. We define $M[X] = (X, D')$ where $D' = \{Y \in D \mid Y \subseteq X\}$, and define $M \setminus X = M[V \setminus X]$. In case $X = \{u\}$ is a singleton, we also write $M \setminus u$ to denote $M \setminus \{u\}$. Set system M is called *proper* if $D \neq \varnothing$. We write simply $Y \in M$ to denote $Y \in D$. A set system M with $\varnothing \in M$ is called *normal* (note that every normal set system is proper). In particular the set system $(\varnothing, \{\varnothing\})$ is normal. A set system M is called *equicardinal* if for all $X_1, X_2 \in M$, $|X_1| = |X_2|$. For convenience we will often simply denote the ground set of the set system under consideration by V.

Let $M = (V, D)$ be a set system. We define, for $X \subseteq V$, *pivot* of M on X, denoted by $M * X$, as $(V, D * X)$, where $D * X = \{Y \oplus X \mid Y \in D\}$. In case $X = \{u\}$ is a singleton, we also write simply $M * u$. Moreover, we define, for $u \in V$, *loop complementation* of M on u (the motivation for this name is from graphs, see Section 9), denoted by $M + u$, as (V, D'), where $D' = D \oplus \{X \cup \{u\} \mid X \in D, u \notin X\}$. We assume left associativity of set system operations. Therefore, e.g., $M + u \setminus u * v$ denotes $((M + u) \setminus u) * v$.

It has been shown in [12] that pivot $*u$ and loop complementation $+u$ on a common element $u \in V$ are involutions (i.e., of order 2) that generate a group F_u isomorphic to S_3, the group of permutations on 3 elements. In particular, we have $+u * u + u = *u + u * u$, which is the third involution (in addition to pivot and loop complementation), and is called the *dual pivot*, denoted by $\bar{*}$. The elements of F_u are called *vertex flips*.[1] We have, e.g., $+u * u = \bar{*} u + u = *u \bar{*} u$ and $*u + u = +u \bar{*} u = \bar{*} u * u$ for $u \in V$ (which are the two vertex flips in F of order 3).

While on a single element the vertex flips behave as the group S_3, they commute when applied on different elements. Hence, e.g., $M * u + v = M + v * u$ and $M \bar{*} u + v = M + v \bar{*} u$ when $u \neq v$. Also, $M + u + v = M + v + u$ and thus we (may) write, for $X = \{u_1, u_2, \ldots, u_n\} \subseteq V$, $M + X$ to denote $M + u_1 \ldots + u_n$ (as the result is independent on the order in which the operations $+u_i$ are applied). Similarly, we define $M \bar{*} X$ for $X \subseteq V$. We will often use the above equalities without explicit mention.

[1] The notion of vertex flip as defined in this paper corresponds to the notion of invertible vertex flip in [12] — as we consider only invertible vertex flips in this paper for notational convenience we omit the adjective "invertible".

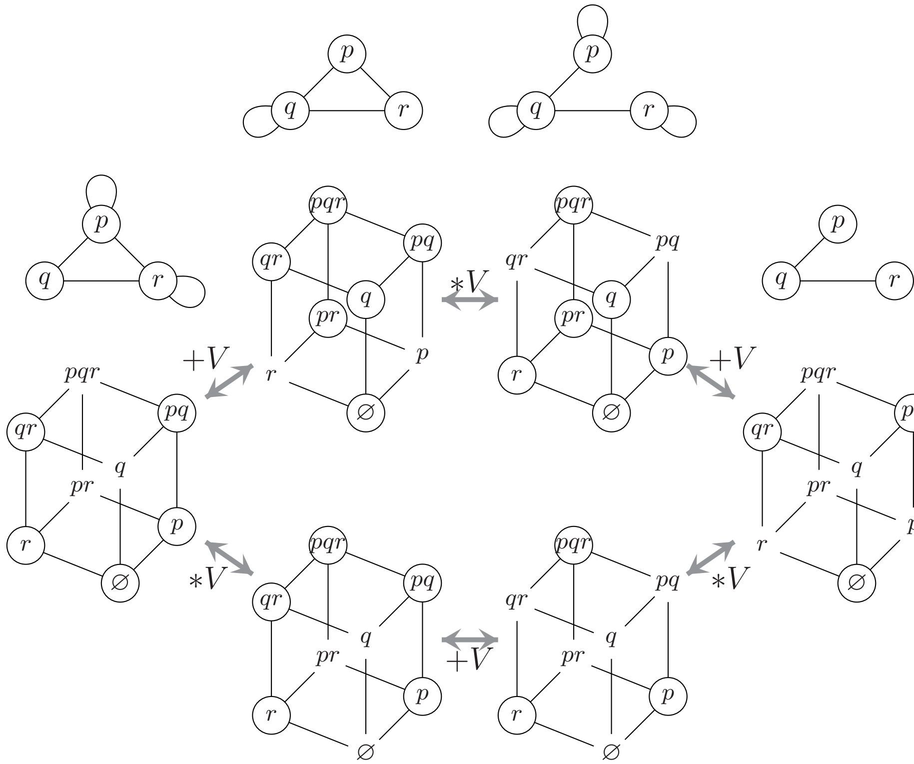

Figure 1: The orbit of a set system under vertex flip on the ground set, cf. Example 1.

One may explicitly define the sets in $M * V$, $M + V$, and $M \mathbin{\bar{*}} V$ as follows: $X \in M * V$ iff $V - X \in M$, and $X \in M + V$ iff $|\{Z \in M \mid Z \subseteq X\}|$ is odd. Dually, $X \in M \mathbin{\bar{*}} V$ iff $|\{Z \in M \mid X \subseteq Z\}|$ is odd. In particular $\varnothing \in M \mathbin{\bar{*}} V$ iff the number of sets in M is odd.

Example 1. Let $V = \{p, q, r\}$. In Figure 1 we show the six sets in the orbit of a set system M under vertex flip over its ground set V. In the Hasse diagram of each set system, the sets that belong to the set system are indicated by a circle, e.g., for the leftmost set system $M = (\ V, \{\ \{p, q\}, \{q, r\}, \{p\}, \{r\}, \varnothing\ \}\ )$.

In Section 9 we learn that actually the topmost four set systems represent graphs, these are indicated in the figure.

It is shown in [12] that vertex flips commute with the removal of different elements from the ground set, and that $M + u \setminus u = M \setminus u$. This is explicitly stated as a lemma.

Lemma 2 ([12])**.** *Let M be a set system and $u, v \in V$ with $u \neq v$. Then $M + u \setminus v = M \setminus v + u$, $M * u \setminus v = M \setminus v * u$, and $M + u \setminus u = M \setminus u$.*

Assume that M is proper. For $X \subseteq V$, we define $d_M(X) = \min(\{|X \oplus Y| \mid Y \in M\})$. Hence, $d_M(X)$ is the smallest distance between X and the sets in

M, where the distance between two sets is measured as the number of elements in the symmetric difference. We set $d_M = d_M(\varnothing)$, the size of a smallest set in M.

Lemma 3 ([11]). *Let M be a proper set system. Then $d_{M*Z}(X) = d_M(X \oplus Z)$ for all $X, Z \subseteq V$, and $d_{M+Y} = d_M$ for all $Y \subseteq V$.*

In particular, we have by Lemma 3, $d_{M*Z} = d_M(Z)$ for all $Z \subseteq V$.

A Δ-*matroid* is a proper set system M that satisfies the *symmetric exchange axiom*: For all $X, Y \in M$ and all $u \in X \oplus Y$, either $X \oplus \{u\} \in M$ or there is a $v \in X \oplus Y$ with $v \neq u$ such that $X \oplus \{u, v\} \in M$ [6]. Note that Δ-matroids are closed under pivot, i.e., $M * X$ for $X \subseteq V$ is a Δ-matroid when M is a Δ-matroid. Also, if $M \setminus u$ with $u \in V$ is proper, then $M \setminus u$ is a Δ-matroid when M is. The next result from [11] shows that Δ-matroids are such that the distance d_M is retained under removal of elements from the ground set.

Proposition 4 ([11]). *Let M be a Δ-matroid, and $X \subseteq V$. If $M[X]$ is proper, then $d_{M[X]} = d_M$.*

We say that M is a *vf-closed* Δ-*matroid* if for any (possibly empty) sequence φ of vertex flips (equivalently, pivots and loop complementations) over V we have that $M\varphi$ is a Δ-matroid. If $M \setminus u$ with $u \in V$ is proper, then $M \setminus u$ is a vf-closed Δ-matroid when M is, see [11].

We assume in this paper that a matroid $M = (V, \beta)$ is described by its set of bases β. Then, a proper set system is an equicardinal Δ-matroid iff it is a matroid [8, Proposition 3]. In this way, the notion of Δ-matroid is a generalization of the notion of matroid. Note that if M is a matroid, then d_M and d_{M*V} are the rank and nullity of M. The following result is shown in [11].

Proposition 5 ([11]). *Every binary matroid is vf-closed.*

Moreover, it is shown in [11] that the 4-point line (a non-binary matroid) is vf-closed, while the 6-point line is not vf-closed.

3. Interlace Polynomial for Set Systems

We establish a, rather generic, multivariate polynomial for set systems, and we obtain a technical result that shows how the variables of the polynomial change when one of the vertex flip operations $+Y$, $\bar{*}Y$ and $*Y$ is applied. In the next section more interesting polynomials appear as specializations.

Let V be a finite set. We define $\mathcal{P}_3(V)$ to be the set of triples (V_1, V_2, V_3) where V_1, V_2, and V_3 are pairwise disjoint subsets of V such that $V_1 \cup V_2 \cup V_3 = V$. Therefore V_1, V_2, and V_3 form an "ordered partition" of V where $V_i = \varnothing$ for some $i \in \{1, 2, 3\}$ is allowed.

Definition 6. Let M be a proper set system. We define the *(multivariate) interlace polynomial* of M as follows:

$$Q(M) = \sum_{(A,B,C) \in \mathcal{P}_3(V)} a_A b_B c_C y^{d_{M*B\,\bar{*}\,C}}.$$

Note that the variables a_A, b_B, and c_C "encode" the partition $(A, B, C) \in \mathcal{P}_3(V)$. Note also that one may write $y^{d_{M+A*B\,\bar{*}\,C}}$ instead of $y^{d_{M*B\,\bar{*}\,C}}$ in the definition of $Q(M)$ (indeed, as A is disjoint from B and C, $+A$ commutes with the other operations and we apply Lemma 3 to obtain $d_{M+A*B\,\bar{*}\,C} = d_{M*B\,\bar{*}\,C}$). In Section 10 we will find that $Q(M)$ generalizes the single-variable case (case $u = 1$) of the multivariate interlace polynomial for simple graphs (as defined in [14]).

The next result shows that the vertex flip operations result in a permutation of the variables a_A, b_B, and c_C.

Theorem 7. *Let M be a proper set system and $Y \subseteq V$. Then*

$$\begin{aligned}
Q(M+Y) &= \sum_{(A,B,C)\in\mathcal{P}_3(V)} a_A b_{(B\oplus Y')} c_{(C\oplus Y')} y^{d_{M*B\,\bar{*}\,C}} \text{ with } Y' = Y \setminus A, \\
Q(M \,\bar{*}\, Y) &= \sum_{(A,B,C)\in\mathcal{P}_3(V)} a_{(A\oplus Y')} b_B c_{(C\oplus Y')} y^{d_{M*B\,\bar{*}\,C}} \text{ with } Y' = Y \setminus B, \text{ and} \\
Q(M*Y) &= \sum_{(A,B,C)\in\mathcal{P}_3(V)} a_{(A\oplus Y')} b_{(B\oplus Y')} c_C y^{d_{M*B\,\bar{*}\,C}} \text{ with } Y' = Y \setminus C.
\end{aligned}$$

Proof. We use that the vertex flips on different vertices commute, and we use the relations between the operations when applied to the same vertex, e.g., $+v*v = \bar{*}v+v$, and $+v\,\bar{*}\,v = *v+v$.

We consider first $Q(M+Y)$. Let $(A, B, C) \in \mathcal{P}_3(V)$. We show the equality $M+Y*B\,\bar{*}\,C = M*(B\oplus Y')\,\bar{*}(C\oplus Y')+Y$ with $Y' = Y \setminus A$, from which $d_{M+Y*B\,\bar{*}\,C} = d_{M*(B\oplus Y')\,\bar{*}(C\oplus Y')}$ follows by Lemma 3.

It suffices to consider $Y = \{v\}$, as the general case follows by iteration. We consider each of the cases $v \in A$, $v \in B$, and $v \in C$ separately.

Assume first that $v \in A$. As $v \notin B$ and $v \notin C$, $M+v*B\,\bar{*}\,C = M*B\,\bar{*}\,C+v$. Assume now that $v \in B$. We have $M+v*B\,\bar{*}\,C = M*(B\oplus\{v\})\,\bar{*}\,C+v*v = M*(B\oplus\{v\})\,\bar{*}\,C\,\bar{*}\,v+v = M*(B\oplus\{v\})\,\bar{*}(C\oplus\{v\})+v$. Finally, assume that $v \in C$. We have $M+v*B\,\bar{*}\,C = M*B\,\bar{*}(C\oplus\{v\})+v\,\bar{*}\,v = M*B\,\bar{*}(C\oplus\{v\})*v+v$, which in turn is equal to $M*(B\oplus\{v\})\,\bar{*}(C\oplus\{v\})+v$.

We have $Q(M+Y) = \sum_{(A,B,C)\in\mathcal{P}_3(V)} a_A b_B c_C y^{d_{M+v*B\,\bar{*}\,C}} = \sum_{(A,B,C)\in\mathcal{P}_3(V)} a_A b_B c_C y^{d_{M*(B\oplus Y')\,\bar{*}(C\oplus Y')}}$ with $Y' = Y\setminus A$. The change of variables $B := B\oplus Y'$ and $C := C\oplus Y'$ leads to a new partition in $\mathcal{P}_3(V)$ (as $Y'\cap A = \varnothing$) and we have $Q(M+Y) = \sum_{(A,B,C)\in\mathcal{P}_3(V)} a_A b_{(B\oplus Y')} c_{(C\oplus Y')} y^{d_{M*B\,\bar{*}\,C}}$. Therefore the result for $Q(M+Y)$ is now verified.

We consider now $Q(M\,\bar{*}\,Y)$. Let $(A, B, C) \in \mathcal{P}_3(V)$. We show the equality $M\,\bar{*}\,Y*B\,\bar{*}\,C = M*B\,\bar{*}(C\oplus Y')+(Y\cap B)$ with $Y' = Y\setminus B$. For $v \in A$ or $v \in C$ the equality $M\,\bar{*}\,v*B\,\bar{*}\,C = M*B\,\bar{*}(C\oplus\{v\})$ is clear. Assume now that $v \in B$. Then $M\,\bar{*}\,v*B\,\bar{*}\,C = M*(B\oplus\{v\})\,\bar{*}\,C\,\bar{*}\,v*v = M*(B\oplus\{v\})\,\bar{*}\,C*v+v = M*B\,\bar{*}\,C+v$. Therefore we obtain, similarly as before, the statement for $Q(M\,\bar{*}\,Y)$.

Finally consider $Q(M*Y)$. Let $(A, B, C) \in \mathcal{P}_3(V)$. We show the equality $M*Y*B\,\bar{*}\,C = M*(B\oplus Y')\,\bar{*}\,C+(Y\cap C)$ with $Y' = Y\setminus C$. For

$v \in A$ or $v \in B$ the equality $M * v * B \mathbin{\bar{*}} C = M * (B \oplus \{v\}) \mathbin{\bar{*}} C$ is clear. Assume now that $v \in C$. Then $M * v * B \mathbin{\bar{*}} C = M * B \mathbin{\bar{*}} (C \oplus \{v\}) * v \mathbin{\bar{*}} v = M * B \mathbin{\bar{*}} (C \oplus \{v\}) \mathbin{\bar{*}} v + v = M * B \mathbin{\bar{*}} C + v$. Therefore we obtain, similarly as before, the statement for $Q(M * Y)$. □

We say that $Q(M)$ is *weighted* if $a_A = \Pi_{u \in A} a_{\{u\}}$ and similarly for $b_B = \Pi_{u \in B} b_{\{u\}}$ and $c_C = \Pi_{u \in C} c_{\{u\}}$. The variables a_A, b_B, and c_C are determined in this way by the values of $a_{\{u\}}$, $b_{\{u\}}$, and $c_{\{u\}}$ for $u \in V$ (note that the empty product $a_\varnothing = b_\varnothing = c_\varnothing$ is equal to 1). For a singleton $\{u\}$ with $u \in V$, we write simply $a_u = a_{\{u\}}$, $b_u = b_{\{u\}}$, and $c_u = c_{\{u\}}$. From now on *we assume $Q(M)$ is weighted*.

4. Specializations of $Q(M)$

In this section we consider four interesting special cases of the multivariate interlace polynomial Q, which generalizes the two types of interlace polynomials [1, 3] and the bracket polynomial [20] for graphs. These special cases each fulfill a particular invariance result w.r.t. pivot, loop complementation, or dual pivot. These invariance results in general do not hold for Q itself. The main reason for introducing Q is to put these polynomials under a common umbrella and to "share" Theorem 7. Finally, we show that the Tutte polynomial on the diagonal can in turn be seen as a special case of one of the specializations of Q.

In this section we again let M be a proper set system.

4.1. Polynomial $Q_1(M)$ invariant under pivot and loop complementation

Let $Q_1(M)$ be the weighted polynomial obtained from $Q(M)$ by the following substitution:

$$[a_u := 1,\ b_u := 1,\ c_u := 1 \text{ for all } u \in V].$$

Therefore, $Q_1(M) = \sum_{X,Y \subseteq V, X \cap Y = \varnothing} y^{d_{M * Y \mathbin{\bar{*}} X}}$.

Lemma 8. *We have*

$$Q_1(M) = \sum_{X \subseteq V} \sum_{Z \subseteq X} y^{d_{M+Z}(X)}.$$

Proof. We have for $X, Y \subseteq V$ with $X \cap Y = \varnothing$, $M * Y \mathbin{\bar{*}} X = M * Y + X * X + X = M + X * Y * X + X = M + X * (X \cup Y) + X$. Moreover, $d_{M+X*(X \cup Y)+X} = d_{M+X*(X \cup Y)} = d_{M+X}(X \cup Y)$. Change variables $[Z := X,\ X := X \cup Y]$. □

By Theorem 7, $Q_1(M)$ is invariant under pivot, loop complementation, and dual pivot. Note that the invariance under any two of these operations implies invariance under the third. We will see in Section 10 that $Q_1(M)$ generalizes an interlace polynomial for simple graphs defined in [1].

We now give polynomials $q_1(M)$, $q_2(M)$, and $q_3(M)$ which are invariant under pivot, loop complementation, and dual pivot, respectively.

4.2. Polynomial $q_1(M)$ invariant under pivot

Let $q_1(M)$ be the polynomial obtained from $Q(M)$ by restricting to partitions $(A, B, \varnothing) \in \mathcal{P}_3(V)$ and removing the partition coding variables a_A and b_B. Formally, we apply the following substitution to $Q(M)$:

$$[a_u := 1,\, b_u := 1,\, c_u := 0 \text{ for all } u \in V].$$

Therefore, $q_1(M) = \sum_{X \subseteq V} y^{d_{M*X}}$. By Theorem 7 (or by the explicit formulation of $q_1(M)$), $q_1(M)$ is invariant under pivot. It follows directly from Lemma 3 that

$$q_1(M) = \sum_{X \subseteq V} y^{d_M(X)}.$$

In Section 10 we will find that $q_1(M)$ generalizes the (single-variable) interlace polynomial (from [3]).

4.3. Polynomial $q_2(M)$ invariant under loop complementation

Let $q_2(M) = q_1(M \mathbin{\bar{*}} V)$. By Theorem 7 the variables a_X and c_X change role, and thus $q_2(M)$ is the polynomial obtained from $Q(M)$ by the following substitution:

$$[a_u := 0,\, b_u := 1,\, c_u := 1 \text{ for all } u \in V].$$

Therefore, $q_2(M) = \sum_{X \subseteq V} y^{d_{M*(V \setminus X) \bar{*} X}}$.

Lemma 9. *We have*

$$q_2(M) = \sum_{X \subseteq V} y^{d_{M+X}(V)}.$$

Proof. The proof is essentially the proof of Lemma 8 for the case $Y = V \setminus X$. In this way, we have for $X \subseteq V$, $d_{M*(V \setminus X) \bar{*} X} = d_{M+X}(V)$. □

By Theorem 7 (or by the explicit formulation in the lemma above), $q_2(M)$ is invariant under loop complementation. We will find that $q_2(M)$ generalizes the bracket polynomial for graphs defined in [20], see Section 10.

4.4. Polynomial $q_3(M)$ invariant under dual pivot

Let $q_3(M) = q_1(M + V)$. By Theorem 7 the variables b_X and c_X change role, and thus $q_3(M)$ is the polynomial obtained from $Q(M)$ by the following substitution:

$$[a_u := 1,\, b_u := 0,\, c_u := 1 \text{ for all } u \in V].$$

Therefore, $q_3(M) = \sum_{X \subseteq V} y^{d_{M \bar{*} X}}$. By Theorem 7 (or by the explicit formulation of $q_3(M)$), $q_3(M)$ is invariant under dual pivot. Moreover, by Theorem 7, we find that $q_3(M) = q_2(M * V)$.

Lemma 10. *We have*

$$q_3(M) = \sum_{X \subseteq V} y^{d_{M+X}(X)}.$$

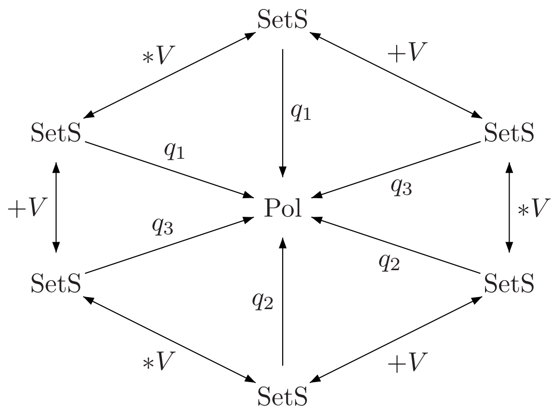

Figure 2: Commutative diagram relating the polynomials q_1, q_2, and q_3.

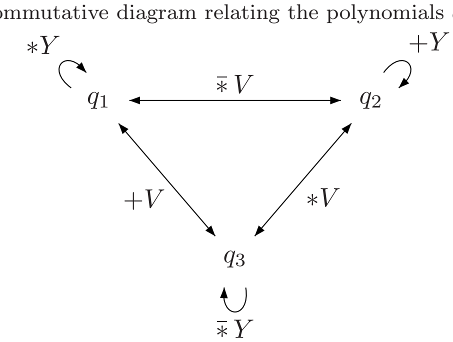

Figure 3: Diagram relating the polynomials q_1, q_2, and q_3. We let $Y \subseteq V$.

Proof. We have $d_{M \,\bar{*}\, X} = d_{M+X*X+X} = d_{M+X*X} = d_{M+X}(X)$. $\square$

Note that since pivot, loop complementation, and dual pivot are involutions, we also have, e.g., $q_2(M) = q_3(M * V)$. The relations between q_1, q_2, and q_3 are illustrated by the commutative diagram of Figure 2, where SetS and Pol denote the family of set systems and family of polynomials, respectively. A more succinct and informal diagram is given in Figure 3.

Example 11. Let $M = (\ V, \{\ \{p,q\}, \{q,r\}, \{p\}, \{r\}, \varnothing\ \}\ )$, cf. Example 1. Then $Q_1(M) = 16 + 10y + y^2$, $q_1(M) = 5 + 3y$, $q_2(M) = q_1(M \,\bar{*}\, V) = 3 + 4y + y^2$, and $q_3(M) = q_1(M + V) = 6 + 2y$.

Note that the invariance properties of Q_1 and q_i ($i \in \{1,2,3\}$) do *not* hold in general for the multivariate polynomial Q itself. In the next sections we will focus on particular recursive relations for Q_1 and q_i ($i \in \{1,2,3\}$), which also do not hold for Q in general.

4.5. Tutte Polynomial on the Diagonal

Let M be a matroid over V (described by its bases). The Tutte polynomial is defined by

$$t_M(x, y) = \sum_{X \subseteq V} (x-1)^{r(V)-r(X)} (y-1)^{n(X)}$$

where $n(X) = \min\{|X \setminus Y| \mid Y \in M\}$ and $r(X) = |X| - n(X)$ are called the *nullity* and *rank* of X in M, respectively. We write now $q_1(M)(y) = q_1(M)$ to explicitly denote the variable y.

Theorem 12. *Let M be a matroid over V. Then $t_M(y, y) = q_1(M)(y-1)$.*

Proof. We have $t_M(y, y) = \sum_{X \subseteq V} (y-1)^{r(V)-r(X)+n(X)}$. It suffices to show that $r(V) - r(X) + n(X) = d_M(X)$. We have $n(X) = \min\{|X \setminus Y| \mid Y \in M\}$. Hence $r(V) - r(X) + n(X) = r(V) - |X| + 2n(X) = \min\{(|Y| - |X| + |X \setminus Y|) + |X \setminus Y| \mid Y \in M\}$ as $r(V)$ is the size of the sets $Y \in M$. Now $|Y| - |X| + |X \setminus Y| = |Y \setminus X|$ and thus we obtain $\min\{(|Y| - |X| + |X \setminus Y|) + |X \setminus Y| \mid Y \in M\} = \min\{|Y \setminus X| + |X \setminus Y| \mid Y \in M\} = \min\{|X \oplus Y| \mid Y \in M\} = d_M(X)$. □

Hence, from Theorem 12 we find that the Tutte polynomial on the diagonal is essentially $q_1(M)$ (note that we may change the variable $y := y - 1$ without loss of generality) for the case where proper set system M is a matroid (described by its bases).

5. Divisibility

The notion of distance in set systems can only be considered for proper set systems. When dealing with recursive relations for set systems we have to verify that both the original set system and the ones after applying operations are proper. This is captured by the notion of divisibility.

We call set system M *divisible by u* if there are $X_1, X_2 \in M$ with $u \in X_1 \oplus X_2$. Therefore, M is divisible by u iff both $M \setminus u$ and $M * u \setminus u$ are proper. We say that M is *divisible* if M is divisible by some $u \in V$. Note that proper M is not divisible iff M contains only one subset. Clearly, M is divisible by u iff $M * u$ is divisible by u. However, if M is divisible by u, then $M + u$ may not be divisible by u — take, e.g., $M = (V, 2^V)$. If φ is any vertex flip on $v \in V$ with $v \neq u$, then M is divisible by u iff $M\varphi$ is divisible by u.

Note that if M is a matroid, then $M \setminus u$ is proper iff u is not a coloop of M, and $M * u \setminus u$ is proper iff u is not a loop of M.

We call M *strongly divisible by u* if M is divisible by u and there is an $X \in M$ with $X \oplus \{u\} \notin M$. Note that there is an $X \in M$ with $X \oplus \{u\} \notin M$ iff $M + u * u \setminus u = M \bar{*} u \setminus u$ is proper. Therefore M is strongly divisible by u iff $M \setminus u$, $M * u \setminus u$, and $M \bar{*} u \setminus u$ are all proper.

Lemma 13. *If M is strongly divisible by u, then for any vertex flip ρ, $M\rho$ is strongly divisible by u.*

Proof. Let M be strongly divisible by u, or equivalently, $M \setminus u$, $M * u \setminus u$, and $M \bar{*} u \setminus u$ are all proper. As the vertex flips on u are generated by $*u$ and $+u$, it suffices to show that $M * u$ and $M + u$ are strongly divisible by u.

Consider first $M * u$. We have that $(M * u) \setminus u$, $(M * u) * u \setminus u = M \setminus u$, and $(M * u) \bar{*} u \setminus u = M \bar{*} u + u \setminus u = M \bar{*} u \setminus u$ are all proper. Hence $M * u$ is strongly divisible by u.

Consider now $M + u$. We have that $(M + u) \setminus u = M \setminus u$, $(M + u) * u \setminus u = M \bar{*} u \setminus u$, and $(M + u) \bar{*} u \setminus u = M * u + u \setminus u = M * u \setminus u$ are all proper. Hence $M + u$ is strongly divisible by u. □

We say that M is *strongly divisible* if M is strongly divisible by some $u \in V$. Note that $M\varphi$ contains only one subset iff $M\varphi$ is not divisible.

Lemma 14. *Let M be a proper set system. Then M is strongly divisible iff for every sequence φ of vertex flips, $M\varphi$ contains at least two sets.*

Proof. If $M\varphi$ contains only one subset, then $M\varphi$ is not divisible and therefore $M\varphi$ is not strongly divisible. By Lemma 13 we see that M is not strongly divisible.

Assume now that M is not strongly divisible. Therefore, for all $u \in V$, at least one of $M \setminus u$, $M * u \setminus u$, and $M \bar{*} u \setminus u$ is not proper. Let $Y \subseteq V$ be such that for all $u \in V$, $u \in Y$ iff $M \setminus u$ and $M * u \setminus u$ are proper. We have thus for all $u \in Y$ that $M \bar{*} u \setminus u$ is not proper. Consider now $M + Y$. As $N + w$ is proper iff N is proper for any set system N and w in the ground set of N, we have that for all $u \in V \setminus Y$, either $M + Y \setminus u = M \setminus u + Y$ or $M + Y * u \setminus u = M * u \setminus u + Y$ is not proper. Moreover, for $u \in Y$, we have that $M + Y * u \setminus u = M \bar{*} u \setminus u + (Y \setminus \{u\})$ is not proper by the definition of Y. Hence for all $u \in V$, either $(M + Y) \setminus u$ or $(M + Y) * u \setminus u$ is not proper. Consequently $M + Y$ is not divisible. Thus, $M + Y$ contains only one subset. □

Remark 15. Alternatively, one can show in the proof of Lemma 14 that $M \bar{*} Y$ contains only one subset (instead of $M + Y$).

We may strengthen Lemma 14: if $X \in M\varphi$ is the only subset of $M\varphi$, then $M\varphi * X$ contains only $\varnothing$. Therefore a proper set system M is not strongly divisible iff there is a sequence φ of vertex flips such that $M\varphi$ contains only $\varnothing$.

6. Recursive Relations and Δ-matroids

In this section we will show that the polynomials considered in Section 4 fulfill a specific recursive relation when restricting to Δ-matroids.

We consider recursive relations that iteratively remove elements of V. Therefore, it will be crucial for our purposes that the removal of an element u (from the ground set) does not change the distance, i.e., $d_M = d_{M\setminus u}$. As this is implied by Δ-matroids, see Proposition 4, and in addition the property of being a Δ-matroid is closed under pivot and removing elements from the ground set, we will assume in this section that M is a Δ-matroid. In the cases where we also consider loop complementation, we often assume that M is a vf-closed Δ-matroid (recall that binary matroids are vf-closed Δ-matroids).

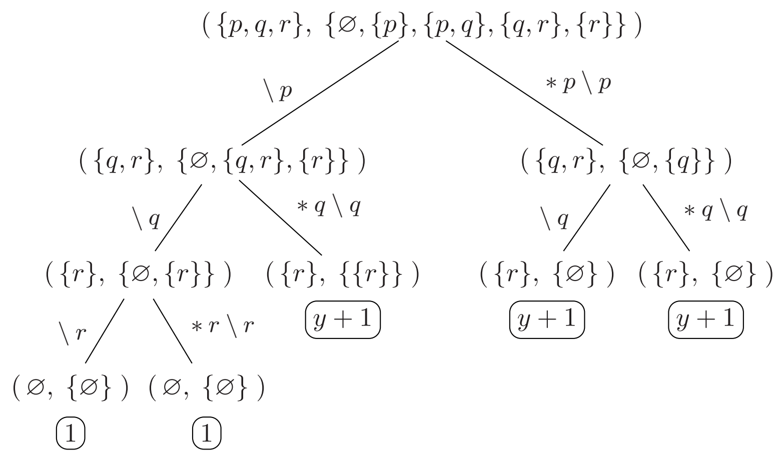

Figure 4: Recursive computation of $q_1(M)$.

6.1. Polynomials $q_i(M)$

We obtain the following recursive relation for $q_1(M)$.

Theorem 16. *Let M be a Δ-matroid, and $u \in V$. If M is divisible by u, then*

$$q_1(M) = q_1(M \setminus u) + q_1(M * u \setminus u).$$

If M is not divisible, then $q_1(M) = (y+1)^n$ with n the size of the ground set of M.

Proof. Assume first that M is divisible by u. Then $q_1(M) = \sum_{X \subseteq V} y^{d_{M*X}} = \sum_{X \subseteq (V \setminus \{u\})} y^{d_{M*X}} + \sum_{X \subseteq (V \setminus \{u\})} y^{d_{M*(X \cup \{u\})}}$. Since M is a Δ-matroid and $M \setminus u$ proper, we have by Proposition 4 $d_{M*X} = d_{M*X \setminus u} = d_{M \setminus u * X}$ and also $d_{M*(X \cup \{u\})} = d_{M*u \setminus u*X}$. Hence $q_1(M) = q_1(M \setminus u) + q_1(M * u \setminus u)$.

If M is not divisible, then M contains only one subset Z. Note now that there are $\binom{n}{i}$ subsets $X \subseteq V$ with distance $i = d_M(X)$ from Z, we have $q_1(M) = (y+1)^n$. $\square$

Note that in Theorem 16, both $M \setminus u$ and $M * u \setminus u$ are Δ-matroids. Hence we may iteratively apply Theorem 16 on the "components" $M \setminus u$ and $M * u \setminus u$ of M to compute $q_1(M)$. Consequently, the equalities of Theorem 16 *characterize* $q_1(M)$.

Example 17. We recursively compute $q_1(M)$ using Theorem 16 for Δ-matroid $M = (\{p, q, r\}, \{\varnothing, \{p\}, \{p, q\}, \{q, r\}, \{r\}\})$ from Example 1. The computation tree is given in Figure 4. Recursion stops when the Δ-matroid is not divisible. We verify that $q_1(M) = 3y + 5$.

We now show that $q_2(M)$ and $q_3(M)$ fulfill similar recursive relations as $q_1(M)$ by applying the previous result to $q_1(M \bar{*} V)$ and $q_1(M + V)$, respectively. As Δ-matroids are not necessarily closed under loop complementation, we explicitly require that $M \bar{*} V$ and $M + V$ are Δ-matroids.

Theorem 18. *Let M be a proper set system and $u \in V$.*
If $M \bar{*} V$ is a Δ-matroid and divisible by u, then

$$q_2(M) = q_2(M * u \setminus u) + q_2(M \bar{*} u \setminus u).$$

If $M + V$ is a Δ-matroid and divisible by u, then

$$q_3(M) = q_3(M \bar{*} u \setminus u) + q_3(M \setminus u).$$

If $M \bar{*} V$ is not divisible, then $q_2(M) = (y+1)^n$ with $n = |V|$. Similarly, If $M + V$ is not divisible, then $q_3(M) = (y+1)^n$.

Proof. Recall that $q_2(M) = q_1(M \bar{*} V)$. Since $M \bar{*} V$ is a Δ-matroid and divisible by u, we have by Theorem 16 $q_1(M \bar{*} V) = q_1(M \bar{*} V \setminus u) + q_1(M \bar{*} V * u \setminus u)$. Now, $q_1(M \bar{*} V \setminus u) = q_1(M \bar{*} u \setminus u \bar{*} (V \setminus \{u\})) = q_2(M \bar{*} u \setminus u)$. Also, $q_1(M \bar{*} V * u \setminus u) = q_1(M \bar{*} u * u \setminus u \bar{*} (V \setminus \{u\})) = q_1(M * u + u \setminus u \bar{*} (V \setminus \{u\})) = q_1(M * u \setminus u \bar{*} (V \setminus \{u\})) = q_2(M * u \setminus u)$.

Recall that $q_3(M) = q_1(M + V)$. Since $M + V$ is a Δ-matroid and divisible by u, we have by Theorem 16 $q_1(M + V) = q_1(M + V \setminus u) + q_1(M + V * u \setminus u)$. We have $q_1(M + V \setminus u) = q_1(M \setminus u + (V \setminus \{u\})) = q_3(M \setminus u)$. Similarly, we have $M + V * u \setminus u = M + u * u \setminus u + (V \setminus \{u\})$, therefore $q_1(M + V * u \setminus u) = q_3(M \bar{*} u \setminus u)$.

If $M \bar{*} V$ is not divisible, then $q_2(M) = q_1(M \bar{*} V) = (y+1)^n$ by Theorem 16. Similarly, if $M + V$ is not divisible, then $q_3(M) = (y+1)^n$. □

Note that $M \bar{*} u$ is divisible by u iff $M \bar{*} V$ is divisible by u. Similarly, $M + u$ is divisible by u iff $M + V$ is divisible by u.

We now consider the case where M is normal, i.e., $\varnothing \in M$. Note that normal M is divisible by u iff there is an $X \in M$ with $u \in X$. Also, by Lemma 3, M is normal iff $M + V$ is normal, and thus in this case $M + V$ is divisible by u iff there is an $Y \in M + V$ with $u \in Y$.

We modify now Theorems 16 and 18 such that each of the "components" of M in the recursive equality are normal whenever M is normal. In this way we prepare for a corresponding result on graphs. For convenience we simply assume M to be vf-closed Δ-matroid (which holds for graphs) instead of assuming $M\varphi$ to be a Δ-matroid for various particular φ as done in Theorem 18. The reader may easily recover this loss of generality.

Corollary 19. *Let M be a normal vf-closed Δ-matroid.*
If $X \in M$ with $u \in X$, then

$$q_1(M) = q_1(M \setminus u) + q_1(M * X \setminus u).$$

If both $\{u, v\} \in M$ and $\{u\}, \{v\} \notin M$, then

$$\begin{aligned} q_2(M) \quad = \quad & q_2(M * \{u,v\} \setminus \{u,v\}) + q_2(M * \{u\} \bar{*} \{v\} \setminus \{u,v\}) + \\ & q_2(M \bar{*} \{u\} \setminus u). \end{aligned}$$

If $Y \in M + V$ with $u \in Y$, then

$$q_3(M) = q_3(M \bar{*} Y \setminus u) + q_3(M \setminus u).$$

Moreover, each of the given "components" is normal.

Proof. Since q_1 is invariant under pivot, $q_1(M * X \setminus u) = q_1(M * \{u\} * (X \setminus \{u\}) \setminus u) = q_1(M * \{u\} \setminus u * (X \setminus \{u\})) = q_1(M * \{u\} \setminus u)$ and the result follows by Theorem 16. Similarly, q_3 is invariant under dual pivot, therefore $q_3(M \bar{*} Y \setminus u) = q_3(M \bar{*} \{u\} \setminus u)$, and the result follows by Theorem 18.

We finally consider q_2. One may easily verify that given $\varnothing, \{u,v\} \in M$ and $\{u\}, \{v\} \notin M$, we have $\varnothing, \{v\}, \{u,v\} \in M \bar{*} \{u\}$ and $\{u\} \notin M \bar{*} \{u\}$. Therefore, $M \bar{*} \{u\}$ is divisible by u. Hence $q_2(M) = q_2(M * \{u\} \setminus u) + q_2(M \bar{*} \{u\} \setminus u)$. As we have $\varnothing, \{v\} \in (M * \{u\} \setminus u) \bar{*} \{v\}$, this set system is divisible by v and therefore $q_2(M) = q_2(M * \{u,v\} \setminus \{u,v\}) + q_2(M * \{u\} \bar{*} \{v\} \setminus \{u,v\}) + q_2(M \bar{*} \{u\} \setminus u)$. We have that $\varnothing$ is contained in each of the components $M * \{u,v\} \setminus \{u,v\}$, $M * \{u\} \bar{*} \{v\} \setminus \{u,v\}$, and $M \bar{*} \{u\} \setminus u$, hence they are all normal. $\square$

6.2. Polynomial $Q_1(M)$

In case M is a normal Δ-matroid, we find that $Q_1(M)$ may be computed given $q_2(N)$ for all "sub-set systems" $N = M[X]$ of M.

Lemma 20. *Let M be a normal Δ-matroid. Then $Q_1(M) = \sum_{X \subseteq V} q_2(M[X])$.*

Proof. Since M is normal, $M[X]$ is proper for all $X \subseteq V$. By Lemma 9 we have therefore $q_2(M[X]) = \sum_{Z \subseteq X} y^{d_{M[X]+Z}(X)}$ for all $X \subseteq V$. As M is a Δ-matroid, we have by Proposition 4 for $Z \subseteq X$, $d_{M[X]+Z}(X) = d_{M+Z[X]}(X) = d_{M+Z}(X)$. Hence by Lemma 8 $Q_1(M) = \sum_{X \subseteq V} q_2(M[X])$. $\square$

Note that if M is a Δ-matroid (not necessarily normal), then Lemma 20 can be applied, for any $Z \in M$, to normal Δ-matroid $M * Z$ to obtain $Q_1(M * Z) = Q_1(M)$ (recall that Q_1 is invariant under pivot).

We show now that $Q_1(M)$ itself fulfills the following recursive relation.

Theorem 21. *Let M be a vf-closed Δ-matroid. If M is strongly divisible by u, then*

$$Q_1(M) = Q_1(M \setminus u) + Q_1(M * u \setminus u) + Q_1(M \bar{*} u \setminus u).$$

If M is not strongly divisible, then $Q_1(M) = (y + 2)^n$, where $n = |V|$.

Proof. Assume first that M is strongly divisible by u. Then we have

$$Q_1(M) = \sum_{(A,B,C) \in \mathcal{P}_3(V \setminus \{u\})} (y^{d_{M*B \bar{*} C}} + y^{d_{M*(B \cup \{u\}) \bar{*} C}} + y^{d_{M*B \bar{*} (C \cup \{u\})}}),$$

by "splitting" each $(A,B,C) \in \mathcal{P}_3(V \setminus \{u\})$ into the cases $(A \cup \{u\}, B, C) \in \mathcal{P}_3(V)$, $(A, B \cup \{u\}, C) \in \mathcal{P}_3(V)$, and $(A, B, C \cup \{u\}) \in \mathcal{P}_3(V)$.

1. Consider the case $(A \cup \{u\}, B, C) \in \mathcal{P}_3(V)$. Since M is a vf-closed Δ-matroid, we have $d_{M*B \bar{*} C} = d_{M*B \bar{*} C \setminus u} = d_{M \setminus u * B \bar{*} C}$. Hence, we obtain $\sum_{(A,B,C) \in \mathcal{P}_3(V \setminus \{u\})} y^{d_{M*B \bar{*} C}} = Q_1(M \setminus u)$.

2. Consider now the case $(A, B \cup \{u\}, C) \in \mathcal{P}_3(V)$. We have $d_{M*(B\cup\{u\})\,\bar{*}\,C} = d_{M*u*B\,\bar{*}\,C}$. Since M is a vf-closed Δ-matroid, we have $d_{M*u*B\,\bar{*}\,C} = d_{M*u*B\,\bar{*}\,C\setminus u} = d_{M*u\setminus u*B\,\bar{*}\,C}$. Hence, we obtain $\sum_{(A,B,C)\in\mathcal{P}_3(V\setminus\{u\})} y^{d_{M*u\setminus u*B\,\bar{*}\,C}}$ $= Q_1(M*u\setminus u)$.
3. Consider finally the case $(A, B, C\cup\{u\}) \in \mathcal{P}_3(V)$. We have $d_{M*B\,\bar{*}(C\cup\{u\})} = d_{M\,\bar{*}\,u*B\,\bar{*}\,C}$. Again, as M is a vf-closed Δ-matroid, we have $d_{M\,\bar{*}\,u*B\,\bar{*}\,C} = d_{M\,\bar{*}\,u\setminus u*B\,\bar{*}\,C}$. Hence, we obtain $Q_1(M\,\bar{*}\,u\setminus u)$.

Therefore, if M is strongly divisible by u, then $Q_1(M) = Q_1(M\setminus u) + Q_1(M*u\setminus u) + Q_1(M\,\bar{*}\,u\setminus u)$.

Finally, assume that M is not strongly divisible. By (the statement below) Lemma 14, there is a sequence φ of vertex flips such that $M' = M\varphi$ contains only $\varnothing$. As $Q_1(M)$ is invariant under vertex flips, we have $Q_1(M) = Q_1(M')$. Since M' contains only $\varnothing$, we have by definition of pivot and dual pivot, $d_{M'*X} = |X|$ and $M'\,\bar{*}\,X = M'$ for all $X \subseteq V$. Therefore

$$Q_1(M') = \sum_{X,Y\subseteq V, X\cap Y=\varnothing} y^{d_{M'\,\bar{*}\,X*Y}} = \sum_{X,Y\subseteq V, X\cap Y=\varnothing} y^{d_{M'*Y}} = \sum_{Y\subseteq V} 2^{n-|Y|} y^{d_{M'*Y}},$$

and moreover

$$Q_1(M') = \sum_{Y\subseteq V} 2^{n-|Y|} y^{|Y|} = \sum_{i\in\{0,\dots,n\}} \binom{n}{i} 2^{n-i} y^i = (y+2)^n.$$

□

Theorem 21 does not hold for Δ-matroids in general. Indeed, consider Δ-matroid $M = (V, 2^V \setminus \{\varnothing\})$ with $V = \{1,2,3\}$. It is shown in [11] that M is not a vf-closed Δ-matroid. We have $Q_1(M) = 13y + 14$, while $Q_1(M\setminus u) + Q_1(M*u\setminus u) + Q_1(M\,\bar{*}\,u\setminus u) = (3y+6) + (y+2)^2 + (y+2)^2 = 2y^2 + 11y + 14$.

Example 22. We recursively compute $Q_1(M)$ using Theorem 21 for vf-closed Δ-matroid $M = (\{p,q,r\}, \{\varnothing, \{p\}, \{p,q\}, \{q,r\}, \{r\}\})$ from Example 1. The computation tree is given in Figure 5, where we have omitted the ground sets for visual clarity. The recursion stops when the vf-closed Δ-matroid is not strongly divisible. We verify that $Q_1(M) = y^2 + 10y + 16$.

6.3. Tutte polynomial

We reformulate now Theorem 16 to obtain a recursive characterization of $q_1(M)$ which can be directly compared to that of the Tutte polynomial (for matroids). To obtain this alternative formulation it suffices to note that if one of the "components" $M\setminus u$ and $M*u\setminus u$ is not proper (for proper set system M), then the distances $d_M(X)$ and $d_M(X\oplus\{u\})$ differ exactly one.

Corollary 23. *Let M be a Δ-matroid. Then*

$$q_1(M) \quad = \quad \begin{cases} (y+1)\, q_1(M\setminus u) & \text{if } M*u\setminus u \text{ is not proper} \\ (y+1)\, q_1(M*u\setminus u) & \text{if } M\setminus u \text{ is not proper} \\ q_1(M\setminus u) + q_1(M*u\setminus u) & \text{otherwise} \end{cases}$$

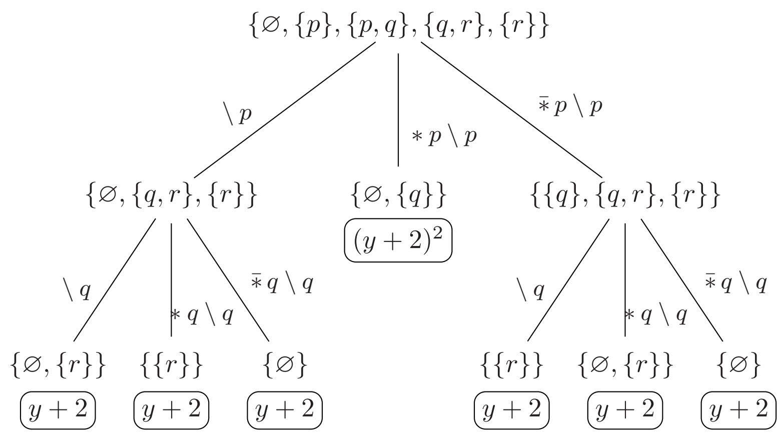

Figure 5: Recursive computation of $Q_1(M)$.

for all $u \in V$*, and* $q_1(M) = 1$ *if* $M = (\varnothing, \{\varnothing\})$.

We recall now the recursive relations of the Tutte polynomial $t_M(x, y)$. Let M be a matroid over V (described by its bases). If $u \in V$ is not a coloop, then *deletion* of matroids coincide with deletion of set systems. If $u \in V$ is not a loop, then *contraction* of M by u, denoted by M/u, is defined as $M/u = (V \setminus \{u\}, \{X \setminus \{u\} \mid u \in X \in M\}) = M * u \setminus u$. Recall that u is a coloop of M iff $M \setminus u$ is not proper, and u is a loop iff $M * u \setminus u$ is not proper. The notion of loop used here is not to be confused with loop complementation and loops in graphs as we use in the rest of the paper.

The Tutte polynomial $t_M(x, y)$ fulfills the following characteristic relations:

$$t_M(x,y) \quad = \quad \begin{cases} y\, t_{M\setminus u}(x,y) & \text{if } u \text{ is a loop} \\ x\, t_{M/u}(x,y) & \text{if } u \text{ is a coloop} \\ t_{M\setminus u}(x,y) + t_{M/u}(x,y) & \text{otherwise} \end{cases}$$

for all $u \in V$, and $t_M(x, y) = 1$ if $M = (\varnothing, \{\varnothing\})$. Hence we find that the recursive relations of $q_1(M)(y-1)$, restricted to matroids M, and $t_M(x, y)$, for the case $x = y$, coincide.

It is not possible to use the recursive setting to obtain a full, two-variable Tutte polynomial for Δ-matroids, as the outcome of the equations would then depend on the order of evaluations. As an example, for Δ-matroid $(\{a, b\}, \{\varnothing, \{a\}, \{a, b\}\})$ the outcome of the recursion might then either be $x + 2$ or $y + 2$.

7. Polynomial Evaluations

We consider now the set system polynomials at particular values of y. Therefore, we will write in this section e.g. $Q_1(M)(y) = Q_1(M)$ to explicitly denote variable y of $Q_1(M)$.

We will see that various results in this section restricted to the case of graphs without loops correspond to known results, see Section 10.

Note that we have $q_i(M)(1) = 2^n$ with $n = |V|$ and $i \in \{1,2,3\}$. Moreover, $Q_1(M)(1) = \sum_{X,Y\subseteq V, X\cap Y=\varnothing} 1 = 3^n$. Also, as $X \in M$ iff $d_M(X) = 0$, we have that $q_1(M)(0)$ is equal to the number of sets in M.

A Δ-matroid M is called *even* if all sets in M are of equal parity.

Theorem 24. *Let M be an even Δ-matroid with $|V| > 0$. Then $q_1(M)(-1) = 0$.*

Proof. As $d_M(Y)$ is the distance of Y to M, the values of $d_{M*X} = d_M(X)$ and $d_{M*X*u} = d_M(X \oplus \{u\})$ cannot differ more than 1 for all $u \in V$ and $X \subseteq V$. As M is even, for d_{M*X} and d_{M*X*u} cannot be equal and thus they differ precisely 1. We have (again) $q_1(M)(y) = \sum_{X\subseteq V} y^{d_{M*X}} = \sum_{X\subseteq (V\setminus\{u\})}(y^{d_{M*X}} + y^{d_{(M*X)*u}})$ for some $u \in V$. Thus, we have $(-1)^{d_{M*X}} + (-1)^{d_{(M*X)*u}} = 0$. □

It is a direct consequence of the recursive relation of the Tutte polynomial that $t_M(0,0) = 0$ for matroid M, except for $M = (\varnothing, \{\varnothing\})$. This property may now be seen as a special case of Theorem 24 (as every matroid is an even Δ-matroid). The result does not hold for Δ-matroids in general: for the Δ-matroid $M = (V, 2^V)$ the polynomial $q_1(M)$ equals (the constant) $2^{|V|}$.

We will need now the following result of [11], which states that the sizes of the minimal sets in M, $M * v$ and $M \bar{*} v$ are closely related.

Proposition 25 ([11])**.** *Let M be a Δ-matroid and $v \in V$. Then the values of d_M, d_{M*v}, and $d_{M\bar{*}v}$ are such that precisely two of the three are equal, to say m, and the third is equal to $m+1$.*

As an immediate consequence we find that $(-2)^{d_M} + (-2)^{d_{M*v}} + (-2)^{d_{M\bar{*}v}} = 2(-2)^m + (-2)^{m+1} = 0$ for a Δ-matroid M and $v \in V$. We will use this fact in the computation of the values of $Q_1(M)(y)$ and $q_1(M)(y)$ at $y = -2$.

Theorem 26. *Let M be a vf-closed Δ-matroid with $|V| > 0$. Then $Q_1(M)(-2) = 0$.*

Proof. Let $n = |V|$. If M is not strongly divisible, then by Theorem 21 $Q_1(M)(-2) = (-2+2)^n = 0$ (as $n > 0$). Let now M be strongly divisible by $u \in V$. We have (again) $Q_1(M)(y) = \sum_{(A,B,C)\in\mathcal{P}_3(V\setminus\{u\})}(y^{d_{M*B\bar{*}C}} + y^{d_{M*(B\cup\{u\})\bar{*}C}} + y^{d_{M*B\bar{*}(C\cup\{u\})}}) = \sum_{(A,B,C)\in\mathcal{P}_3(V\setminus\{u\})}(y^{d_{M*B\bar{*}C}} + y^{d_{(M*B\bar{*}C)*u}} + y^{d_{(M*B\bar{*}C)\bar{*}u}})$. As $M * B \bar{*} C$ is a Δ-matroid for all $B, C \subseteq V$, the result follows by Proposition 25. □

Note that for the vf-closed Δ-matroid $M = (\varnothing, \{\varnothing\})$ we have $Q_1(M)(-2) = 1$. Also, note that for the (not vf-closed) Δ-matroid M considered below Theorem 21, we have $Q_1(M)(-2) = -12$.

We consider now the evaluation of $q_1(M)$ at value -2.

Theorem 27. *Let M be a vf-closed Δ-matroid and $n = |V|$. Then $q_1(M)(-2) = (-1)^n(-2)^{d_{M\bar{*}V}}$.*

Proof. If $n = 0$, we have $M = (\varnothing, \{\varnothing\})$ and therefore $q_1(M)(-2) = 1$ as required. If $n > 0$, then we have (again) $q_1(M)(y) = \sum_{X \subseteq V} y^{d_{M*X}} = \sum_{X \subseteq (V \setminus \{u_1\})} (y^{d_{M*X}} + y^{d_{(M*X)*u_1}})$. By Proposition 25, $(-2)^{d_{M*X}} + (-2)^{d_{(M*X)*u_1}} = (-1)(-2)^{d_{M*X \bar{*} u_1}} = (-1)(-2)^{d_{(M \bar{*} u_1)*X}}$. Hence $q_1(M)(-2) = (-1) \sum_{X \subseteq (V \setminus \{u_1\})} (-2)^{d_{(M \bar{*} u_1)*X}}$. We can repeat this argument for $M \bar{*} u_1$ and $u_2 \in (V \setminus \{u_1\})$ etc. to obtain $q_1(M)(-2) = (-1)^n (-2)^{d_{M \bar{*} V}}$. $\square$

The result of Theorem 27 can be extended to the value $q_1(M)(p-2)$ for any non-zero even integer p when computing modulo p. The proof of this result is slightly more involved than the proof of Theorem 27 as we require additional bookkeeping to distinguish between factors -1 and 1 modulo p. As usual, we write for a negative integer p simply $x \pmod p$ to denote $x \pmod{-p}$.

Theorem 28. *Let M be a vf-closed Δ-matroid and $n = |V|$. Then for every non-zero even integer p, we have $q_1(M)(p-2) = k(-2)^{d_{M \bar{*} V}}$, where k is an integer that satisfies $k \equiv (-1)^n \pmod p$. In particular, k is odd.*

Proof. The proof is by induction on the number ℓ of sets in a vf-closed Δ-matroid. We have $\ell \geq 1$ as M is proper. If $\ell = 1$, then M is not divisible, and we obtain $q_1(M)(p-2) = (p-1)^n \equiv (-1)^n \pmod p$ by Theorem 16. As the number of sets in M is odd, we have $d_{M \bar{*} V} = 0$, and the result holds.

If $\ell \geq 2$, then M is divisible by some $u \in V$. Let $V' = V \setminus \{u\}$. We have by Theorem 16, $q_1(M) = q_1(M \setminus u) + q_1(M * u \setminus u)$. According to the induction hypothesis $q_1(M \setminus u)(p-2) = k_1(-2)^{d_{M \setminus u \bar{*} V'}}$ and $q_1(M * u \setminus u)(p-2) = k_2(-2)^{d_{M*u \setminus u \bar{*} V'}}$, for some k_1 and k_2 with $k_1 \equiv k_2 \equiv (-1)^{n-1} \pmod p$.

By Lemma 2 and Proposition 4 we have that $d_{M \setminus u \bar{*} V'} = d_{M \bar{*} V'}$ and $d_{M*u \setminus u \bar{*} V'} = d_{M \bar{*} V' * u}$. Hence, $q_1(M)(p-2) = k_1(-2)^{d_{M \bar{*} V'}} + k_2(-2)^{d_{M \bar{*} V' * u}}$.

By Proposition 25 the three values $d_{M \bar{*} V'}$, $d_{M \bar{*} V' * u}$ and $d_{M \bar{*} V}$ are of the form m, m, $m+1$, in some order.

First, consider the case where $m = d_{M \bar{*} V'} = d_{M \bar{*} V' * u}$. Let $k = (k_1 + k_2)/(-2)$. As p is even, k is an integer. We have $q_1(M)(p-2) = k_1(-2)^m + k_2(-2)^m = k(-2)^{m+1} = k(-2)^{d_{M \bar{*} V}}$ as required, while also $k \equiv (2(-1)^{n-1})/(-2) = -(-1)^{n-1} = (-1)^n \pmod p$.

Finally, consider the case where $d_{M \bar{*} V'}$ and $d_{M \bar{*} V' * u}$ are of the form m, $m+1$ in some order. Let $k = k_1 - 2k_2$, then $q_1(M)(p-2) = k_1(-2)^m + k_2(-2)^{m+1} = k(-2)^m = k(-2)^{d_{M \bar{*} V}}$ as required, while also $k \equiv (-1)^{n-1} - 2(-1)^{n-1} = -(-1)^{n-1} = (-1)^n \pmod p$. $\square$

By Theorem 27 and Theorem 28, we find that for M and p as above, $q_1(M)(p-2) = k q_1(M)(-2)$ where $k \equiv 1 \pmod p$.

Since $q_2(M) = q_1(M \bar{*} V)$ and $q_3(M) = q_1(M + V)$, we have the following corollary.

Corollary 29. *Let M be a vf-closed Δ-matroid and $n = |V|$. Then $q_2(M)(-2) = (-1)^n(-2)^{d_M}$, $q_3(M)(-2) = (-1)^n(-2)^{d_{M*V}}$, and for every non-zero even integer p, $q_2(M)(p-2) = k(-2)^{d_M}$ and $q_3(M)(p-2) = k'(-2)^{d_{M*V}}$ with $k \equiv k' \equiv (-1)^n \pmod p$.*

By Theorem 12 we may apply Theorems 27 and 28 to obtain new evaluations for the Tutte polynomial. We say that a matroid M is *vf-closed* if M is vf-closed as a Δ-matroid.

Corollary 30. *Let M be a vf-closed matroid and $n = |V|$. Then $t_M(-1,-1) = (-1)^n(-2)^{d_{M \bar{*} V}}$ and for every non-zero even integer p, $t_M(p-1, p-1) = k(-2)^{d_{M \bar{*} V}}$ with $k \equiv (-1)^n \pmod{p}$.*

It is shown in [7] that for a binary matroid M, $t_M(3,3) = kt_M(-1,-1)$ with k an odd integer. As every vf-closed matroid is binary, this is implied by Corollary 30. Furthermore, Conjecture 4.2 of [16] postulates that for all integers p, $t_M(4p-1, 4p-1) = kt_M(-1,-1)$ with k an odd integer (and again M a binary matroid). Corollary 30 proves this conjecture (in fact, this conjecture is a special case of Corollary 30).

In [18] it is shown that $t_M(-1,-1) = (-1)^n(-2)^{\dim(C \cap C^\perp)}$, where C is the cycle space of binary matroid M and $C^\perp$ is the cocycle space of M (i.e., the orthogonal complement of C). The space $C \cap C^\perp$ is called the *bicycle space* of M. Since every binary matroid is vf-closed, by combining Corollary 30 and the result from [18], we obtain the following.

Theorem 31. *Let M be a binary matroid (described by its bases). Then $d_{M \bar{*} V}$ is equal to the dimension of the bicycle space of M, i.e., $d_{M \bar{*} V} = \dim(C \cap C^\perp)$ where C is the cycle space of M.*

Hence, Corollary 30 can thus also be seen as a generalization of that result from [18]. Note that for binary matroid M, the nullity d_{M*V} of M is equal to the dimension of the cycle space, and the rank d_M of M is equal to the dimension of the cocycle space. This illustrates a tight connection between a binary matroid M, its dual matroid $M^* = M * V$, and its bicycle matroid (the matroid corresponding to the bicycle space of M).

8. Pivot on Matrices

In order to interpret our findings on graphs we recall in this and the next section the necessary notions and results from the literature. In this section we take a look at the principal pivot transform operation on matrices. As we recall, this matrix operation corresponds to pivot operation on set systems if the set system represents the nonsingular submatrices of the matrix.

For a $V \times V$-matrix A (the columns and rows of A are indexed by finite set V) and $X \subseteq V$, $A[X]$ denotes the principal submatrix of A w.r.t. X, i.e., the $X \times X$-matrix obtained from A by restricting to rows and columns in X. We also define $A \setminus X = A[V \setminus X]$. In case $X = \{u\}$ is a singleton, we also write $A \setminus u$ to denote $A \setminus \{u\}$.

Let $X \subseteq V$ be such that $A[X]$ is nonsingular, i.e., $\det A[X] \neq 0$. The operation *principal pivot transform* (PPT or *pivot* for short) of A on X, denoted

by $A * X$, is defined as follows. Let $A = \begin{pmatrix} P & Q \\ R & S \end{pmatrix}$ with $P = A[X]$. Then

$$A * X = \begin{pmatrix} P^{-1} & -P^{-1}Q \\ RP^{-1} & S - RP^{-1}Q \end{pmatrix}.$$

The PPT operation is essentially from [22]. The principal submatrix $(A * X) \setminus X = S - RP^{-1}Q$ in the lower-right corner of $A*X$ is called the *Schur complement* of X in A.

The pivot can be considered a partial inverse, as A and $A * X$ satisfy the following characteristic relation, where the vectors x_1 and y_1 correspond to the elements of X.

$$A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \text{ iff } A * X \begin{pmatrix} y_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ y_2 \end{pmatrix} \tag{1}$$

Equality (1) can be used to define $A*X$ given A and X: any matrix B satisfying this equality is of the form $B = A * X$, see [21, Theorem 3.1], and therefore such B exists precisely when $A[X]$ is nonsingular. Note that if A is nonsingular, then $A * V = A^{-1}$. Also note by Equation (1) that a pivot operation is an involution (operation of order 2), and more generally, if $(A * X) * Y$ is defined, then $A * (X \oplus Y)$ is defined and they are equal.

By convention, the nullity of the empty matrix is 0, i.e., $n(A[\varnothing]) = 0$.

For a $V \times V$-matrix A we consider the associated set system $\mathcal{M}_A = (V, D_A)$ with $D_A = \{X \subseteq V \mid \det A[X] \neq 0\}$. Moreover as observed in [6] pivot on matrices and on set systems coincide through the equality $\mathcal{M}_{A*X} = \mathcal{M}_A * X$ for $X \subseteq V$ when the left-hand side is defined.

It follows from the definition of pivot that $A*X$ is skew-symmetric whenever A is. Moreover, if A is skew-symmetric, then $\mathcal{M}_A$ is a Δ-matroid (see [6]) and we have the following correspondence between distance and nullity: $d_{\mathcal{M}_A}(X) = n(A[X])$ for $X \subseteq V$ (see [11]).

9. Pivot and Loop Complementation on Graphs

In this section we recall results on pivot and loop complementation on graphs.

We consider now undirected graphs without parallel edges, but we do allow loops. For graph $G = (V, E)$ we use $V(G)$ and $E(G)$ to denote its set of vertices V and set of edges E, respectively, and for $x \in V$, $\{x\} \in E$ iff x has a loop.

With a graph G one associates its adjacency matrix $A(G)$, which is a $V \times V$-matrix $(a_{u,v})$ over $\mathbb{F}_2$ with $a_{u,v} = 1$ iff $\{u, v\} \in E$ (we have $a_{u,u} = 1$ iff $\{u\} \in E$). In this way, the family of graphs with vertex set V corresponds precisely to the family of symmetrical $V \times V$-matrices over $\mathbb{F}_2$. Therefore we often make no distinction between a graph and its matrix, so, e.g., by the nullity of graph G, denoted $n(G)$, we mean the nullity $n(A(G))$ of its adjacency matrix (computed over $\mathbb{F}_2$). Also, we (may) write, e.g., $\mathcal{M}_G = \mathcal{M}_{A(G)}$, $G[X] = A(G)[X]$ (for $X \subseteq V$), and $G \setminus u = A(G) \setminus u$ (for $u \in V$). Note that $G[X]$ is the subgraph of

G induced by X. Similar as for set systems, we often simply write V to denote the vertex set of the graph under consideration.

Given $\mathcal{M}_G$ for some graph G, one can (re)construct the graph G: $\{u\}$ is a loop in G iff $\{u\} \in \mathcal{M}_G$, and $\{u,v\}$ is an edge in G iff $(\{u,v\} \in \mathcal{M}_G) \oplus ((\{u\} \in \mathcal{M}_G) \wedge (\{v\} \in \mathcal{M}_G))$, see [10, Property 3.1]. Hence the function $\mathcal{M}_{(\cdot)}$ which assigns to each graph G its set system $\mathcal{M}_G$ is injective. In this way, the family of graphs (with set V of vertices) can be considered as a subset of the family of set systems (over set V).

For a graph G and a set $X \subseteq V$, the graph obtained after *loop complementation* for X on G, denoted by $G + X$, is obtained from G by adding loops to vertices $v \in X$ when v does not have a loop in G, and by removing loops from vertices $v \in X$ when v has a loop in G. Hence, if one considers a graph as a matrix, then $G + X$ is obtained from G by adding the $V \times V$-matrix with elements $x_{i,j}$ such that $x_{i,i} = 1$ if $i \in X$ and 0 otherwise. Note that $(G+X)+Y = G+(X \oplus Y)$. It has been shown in [12] that $\mathcal{M}_{G+X} = \mathcal{M}_G + X$ for $X \subseteq V$.

Note that a graph G is skew-symmetric over $\mathbb{F}_2$. Therefore by Section 8 $G*X$ is skew-symmetric (over $\mathbb{F}_2$) as well, and thus $G * X$ is also a graph. Moreover, by Section 8 $\mathcal{M}_{G*X} = \mathcal{M}_G * X$ for $X \subseteq V$ if the left-hand side is defined. It is shown in [11] that $\mathcal{M}_G$ is a vf-closed Δ-matroid when G is a graph.

The pivots $G * X$ where X is a minimal set of $\mathcal{M}_G \backslash \{\varnothing\}$ (the set system obtained from $\mathcal{M}_G$ by removing $\varnothing$) w.r.t. inclusion are called *elementary*. It is noted in [15] that an elementary pivot X corresponds to either a loop, $X = \{u\} \in E(G)$, or to an edge, $X = \{u,v\} \in E(G)$, where (distinct) vertices u and v are both non-loops. Any pivot on a graph can be decomposed into a sequence of elementary pivots. The elementary pivot $G * \{u\}$ on a loop $\{u\}$ is called *local complementation*, and the elementary pivot $G * \{u,v\}$ on an edge $\{u,v\}$ between non-loop vertices is called *edge complementation*. Local complementation "complements" the edges in the neighbourhood $N_G(u) = \{v \in V \mid \{u,v\} \in E(G), u \neq v\}$ of u in G: for each $v,w \in N_G(u)$, $\{v,w\} \in E(G)$ iff $\{v,w\} \notin E(G * \{u\})$, and $\{v\} \in E(G)$ iff $\{v\} \notin E(G * \{u\})$ (the case $v = w$). The other edges are left unchanged. We will not recall (or use) the explicit graph theoretical definition of edge complementation in this paper. It can be found in, e.g., [12]. Similar as for pivot we may define, e.g., "dual local complementation" $G \bar{*} \{u\} = G + \{u\} * \{u\} + \{u\}$ which is identical to "regular" local complementation, except that it is defined for a non-loop $\{u\}$ (instead of a loop).

For convenience, the next proposition summarizes the key known results of the previous section and this section which we will use frequently in the remaining part of this paper.

Proposition 32. *Let G be a graph and $X \subseteq V$. Then the normal vf-closed Δ-matroid $\mathcal{M}_G$ uniquely determines G (and the other way around). Moreover, $\mathcal{M}_{G*X} = \mathcal{M}_G * X$ (if the left-hand side is defined), $\mathcal{M}_{G+X} = \mathcal{M}_G + X$, and $d_{\mathcal{M}_G}(X) = n(G[X])$.*

10. Graph Polynomials

We now turn to graphs and reinterpret our results on interlace polynomials for Δ-matroids in this domain. We consider now $q_i(M)$ for $i \in \{1,2,3\}$, $Q_1(M)$, and $Q(M)$ for the case $M = \mathcal{M}_G$ for some graph G. For notational convenience we denote them by $q_i(G)$, $Q_1(G)$, and $Q(G)$.

We obtain by Section 4 and by Proposition 32 the following graph polynomials.

$$q_1(G) = \sum_{X\subseteq V} y^{n(G[X])} = \sum_{X\subseteq V} y^{n(G\setminus X)}, \tag{2}$$

$$q_2(G) = \sum_{X\subseteq V} y^{n(G+X)}, \tag{3}$$

$$q_3(G) = \sum_{X\subseteq V} y^{n(G+V[X])} \tag{4}$$

Polynomial $q_1(G)$ is the *interlace polynomial* $q(G)$ as defined in [3] (modulo the irrelevant change of variables $y := y-1$). Moreover polynomial $q_2(G)(y)$ is equal to the *bracket polynomial* [2] $b_B(G)(y) = \sum_{X\subseteq V} B^{|X|} y^{n(G+X)}$ for graphs as defined in [20] for the case $B = 1$.

Therefore the definitions of $q_1(M)$ and $q_2(M)$ for arbitrary set system M can be seen as a generalization of the interlace polynomial and the (restricted) bracket polynomial. From this point of view we notice a close similarity between the two polynomials, as shown in the next result (which follows directly from the definitions of $q_2(M)$ and $q_3(M)$).

Theorem 33. *Let G be a graph. We have (1) $q_1(G \bar{*} V) = q_2(G)$, $q_1(G+V) = q_3(G)$, $q_2(G * V) = q_3(G)$, assuming $G * V$ and $G \bar{*} V$ are defined, and (2) for $Y \subseteq V$, $q_1(G * Y) = q_1(G)$, $q_2(G+Y) = q_2(G)$, $q_3(G \bar{*} Y) = q_3(G)$, assuming $G * Y$ and $G \bar{*} Y$ are defined.*

We now formulate the recursive relations from Section 6 concerning the $q_i(M)$ for normal set systems M for the case that M represents a graph G.

Theorem 34. *Let G be a graph.*

- *Let $X \subseteq V$ such that $u \in X$ and $G * X$ is defined, then*
$$q_1(G) = q_1(G \setminus u) + q_1(G * X \setminus u),$$
- *If both $\{u,v\}$ is an edge of G where both u and v do not have loops, then*
$$\begin{aligned} q_2(G) &= q_2(G * \{u,v\} \setminus \{u,v\}) + q_2(G \bar{*} \{v\} * \{u\} \setminus \{u,v\}) + \\ &\quad q_2(G \bar{*} \{u\} \setminus u) \\ &= q_2(G * \{u,v\} \setminus \{u,v\}) + q_2(G * \{u,v\} \bar{*} \{v\} \setminus \{u,v\}) + \\ &\quad q_2(G \bar{*} \{u\} \setminus u). \end{aligned}$$

[2] Actually, the bracket polynomial contains another variable A. However, as pointed out in [20] we may assume without loss of information either $A = 1$ or $B = 1$ (not both) — for convenience we choose $A = 1$.

- *Let $Y \subseteq V$ such that $u \in Y$ and $G \bar{*} Y$ is defined, then*

$$q_3(G) = q_3(G \setminus u) + q_3(G \bar{*} Y \setminus u).$$

Proof. The recursive relations for $q_1(G)$ and $q_3(G)$ follow directly from Corollary 19.

We now consider $q_2(G)$. It is easy to see that $G * \{u, v\}$, $G \bar{*}\{v\} * \{u\}$, and $G \bar{*}\{u\}$ are defined. Thus the equality $q_2(G) = q_2(G*\{u, v\} \setminus \{u, v\}) + q_2(G \bar{*}\{v\} * \{u\} \setminus \{u, v\}) + q_2(G \bar{*}\{u\} \setminus u)$ follows now from Corollary 19. Moreover we have $\mathcal{M}_G * \{u, v\} \bar{*}\{v\} \setminus \{u, v\} = \mathcal{M}_G * \{u\} * \{v\} \bar{*}\{v\} \setminus \{u, v\} = \mathcal{M}_G * \{u\} \bar{*}\{v\} + \{v\} \setminus \{u, v\} = \mathcal{M}_G * \{u\} \bar{*}\{v\} \setminus \{u, v\} = \mathcal{M}_G \bar{*}\{v\} * \{u\} \setminus \{u, v\}$. As it is easy to see that $G * \{u, v\} \bar{*}\{v\}$ is defined, and so we have $G * \{u, v\} \bar{*}\{v\} \setminus \{u, v\} = G \bar{*}\{v\} * \{u\} \setminus \{u, v\}$. Consequently, we obtain the other equality $q_2(G) = q_2(G * \{u, v\} \setminus \{u, v\}) + q_2(G * \{u, v\} \bar{*}\{v\} \setminus \{u, v\}) + q_2(G \bar{*}\{u\} \setminus u)$. □

The recursive relation for the single-variable variant $q_N(G; y)$ of the interlace polynomial in [4, Section 4] is the special case of the recursive relation for $q_1(G)$ in Theorem 34 where the pivot $G * X$ is elementary. Note that the recursive relation for $q_1(M)$ where M is a Δ-matroid is a generalization of this result. Also, it is shown in [13] that the recursive relation for $q_1(G)$ in Theorem 34 can be generalized for arbitrary $V \times V$-matrices A. In this way we find that $q_1(M)$ for Δ-matroids and $q_1(A)$ for matrices are "incomparable" generalizations (in the sense that one is not more general than the other) of the graph polynomial $q_1(G)$.

Of the two recursive relations for q_2 given in Theorem 34, the former seems novel, while the latter is from [20, Theorem 1*(ii)*].

Remark 35. The *marked-graph bracket polynomial* [19] for a graph G is defined as $\mathrm{mgb}_B(G, C) = \sum_{X \subseteq V} B^{|X|} y^{n(G+X[X \cup C])}$ with $C \subseteq V$ (the elements of C are the vertices of G that are *not* marked). For the case $B = 1$, we have that $\mathrm{mgb}_B(G, C) = q_2(G*(V \setminus C)) = q_3(G*C)$. Indeed, $q_3(M*C) = \sum_{X \subseteq V} y^{d_{M*C \bar{*} X}}$, and $d_{M*C \bar{*} X} = d_{M*(C \setminus X)*(C \cap X) \bar{*} X} = d_{M*(C \setminus X) \bar{*} X + (C \cap X)} = d_{M*(C \setminus X) \bar{*} X} = d_{M*(C \setminus X)+X*X} = d_{M+X*(C \setminus X)*X} = d_{M+X*(X \cup C)} = d_{M+X}(X \cup C)$ and so $\mathrm{mgb}_1(G, C)$ is $q_3(M * C)$ where set system M represents graph G (i.e., $M = \mathcal{M}_G$). Therefore $\mathrm{mgb}_1(G, C)$ can be seen as a "hybrid" polynomial "between" q_2 and q_3. The recursive relations for $\mathrm{mgb}_B(G, C)$ deduced in [19] are straightforwardly deduced from the recursive relations for $q_i(G)$. Of course, one may also consider the hybrid polynomials $q_1(G \bar{*} C) = q_2(G \bar{*}(V \setminus C))$ and $q_1(G + C) = q_3(G + V \setminus C)$ for $C \subseteq V$ between $q_1(G)$ and $q_2(G)$, and between $q_1(G)$ and $q_3(G)$, respectively.

We now consider polynomial $Q_1(G)$. We have

$$Q_1(G) = \sum_{X \subseteq V} \sum_{Z \subseteq X} y^{n(G+Z[X])} \tag{5}$$

By Lemma 20 we have $Q_1(G) = \sum_{X \subseteq V} q_2(G[X])$. By Theorem 21 and the fact that $Q_1(M)$ is invariant under pivot and loop complementation, we have the following result.

Theorem 36. *Let G be a graph, and $Y \subseteq V$. We have $Q_1(G) = Q_1(G * Y)$ when $G * Y$ is defined, and $Q_1(G) = Q_1(G + Y)$. If $\{u, v\}$ is an edge in G with $u \neq v$ where both u and v are non-loop vertices, then*

$$Q_1(G) = Q_1(G \setminus u) + Q_1(G \bar{*} \{u\} \setminus u) + Q_1(G * \{u, v\} \setminus u). \tag{6}$$

Note that the recursive relation of Theorem 36 above may be easily modified for the cases where either u or v (or both) have a loop. For example, if u has a loop and v does not have a loop in G, then, since $Q_1(G) = Q_1(G + \{u\})$, we have $Q_1(G) = Q_1(G \setminus u) + Q_1(G * \{u\} \setminus u) + Q_1(G + \{u\} * \{u, v\} \setminus u)$ (as $G + \{u\} \bar{*} \{u\} \setminus u = G * \{u\} \setminus u$).

As $Q_1(G)$ is invariant under loop complementation, we may, as a special case, consider this polynomial for *simple graphs* F. In this way we obtain the recursive relation

$$Q_1(F) = Q_1(F \setminus u) + Q_1(\mathrm{loc}_u(F) \setminus u) + Q_1(F * \{u, v\} \setminus u), \tag{7}$$

where $\mathrm{loc}_u(F)$ is the operation that complements the neighbourhood of u in F without introducing loops (as F is simple, no loops are removed).

Polynomial $Q'(F) = \sum_{X \subseteq V} \sum_{Z \subseteq X} (y - 2)^{n(F + Z[X])}$ has been considered in [1] for simple graphs F[3]. As the difference in variable $y := y - 2$ in $Q_1(G)$ w.r.t. $Q'(F)$ is irrelevant, this polynomial is essentially $Q_1(G)$ restricted to simple graphs. The Equation 7 is shown in [1] from a matrix point-of-view. Similarly, the single-variable case (case $u = 1$) of the multivariate interlace polynomial $C(F)$ of [14] is (essentially) $Q(G)$ restricted to simple graphs.

Finally, we may state the graph analogs of the results of Section 7.

Theorem 37. *Let G be a graph, and $n = |V|$. Then*

$$\begin{aligned} Q_1(G)(-2) &= 0 \quad \text{if } n > 0 & (8) \\ q_1(G)(-2) &= (-1)^n (-2)^{n(G+V)} & (9) \\ q_2(G)(-2) &= (-1)^n & (10) \\ q_1(G)(-1) &= 0 \quad \text{if } n > 0 \text{ and } G \text{ has no loops} & (11) \end{aligned}$$

Moreover, for every non-zero even integer p,

$$q_1(G)(p-2) = k(-2)^{n(G+V)} \tag{12}$$

with $k \equiv (-1)^n \pmod{p}$.

Proof. Equality 8 follows directly from Theorem 26. As $d_{M \bar{*} V} = d_{M+V*V+V} = d_{M+V*V} = d_{M+V}(V)$, Equality 9 follows from Theorem 27. By the paragraph below Theorem 27 we have as $n((G+V)[\varnothing]) = 0$, also $q_2(G)(-2) = (-1)^n$. The equality for $q_1(G)(p-2)$ follows from Theorem 28. Finally, we have that $\mathcal{M}_G$ is an even Δ-matroid iff G has no loops. By Theorem 24, $q_1(G)(-1) = 0$ if G has no loops. □

[3] To ease comparison, the formulation of the definition of $Q'(F)$ is considerably modified w.r.t. to the original formulation in [1] to define $Q'(F)$.

Note that since $q_2(G)(-2) = (-1)^n$ and $q_2(G)$ is a polynomial, we have that $q_2(G)(p-2) \equiv (-1)^n \pmod{p}$. Hence the graph analog of $q_2(M)(p-2)$ in Corollary 29 trivially holds (given the equality of $q_2(G)(-2)$).

Equality 8 may also be deduced from [7, Section 2] through the fundamental graph representation of isotropic systems (and using that $Q_1(G)$ is invariant under loop complementation) — the result in [7] is in turn a generalization of a result on the Martin polynomial.

Equality 9 is proven both in [1, Theorem 2] and in [5, Theorem 1] for the case where G does not have loops. In fact in both these proofs one may recognize a type of $m, m, m+1$ result for matrices comparable to our Proposition 25 for Δ-matroids.

Equality 11 is mentioned in [1, Remark before Lemma 2]. Note that Equality 11 does *not* in general hold for graphs with loops. Indeed, e.g., for the graph G having exactly one vertex u, where u is looped, we have $q_1(G)(-1) = 2$.

It is shown in [1, Theorem 2] that $Q_1(G)(2) = k2^n$ where k is the number of induced Eulerian subgraphs. This result cannot be generalized to vf-closed Δ-matroids as for vf-closed Δ-matroid $M = (\{q,r,s\}, \{\{q\}, \{r\}, \{s\}, \{q,r\}, \{q,s\}, \{r,s\}\})$ we have $Q_1(M) = 9 \cdot (y+2)$. Consequently $Q_1(M)(2) = 9 \cdot 2^2$ is not of the form $k \cdot 2^n$.

Finally, a consequence of the proof of [6, Theorem 4.4] is that if M is a binary matroid, then $M = \mathcal{M}_G$ for some bipartite graph G (hence G has no loops). Conversely, if G is a bipartite graph with color classes X and $V \setminus X$, then $M_G * X$ is a binary matroid (and also $M_G * (V \setminus X)$). Hence we obtain the following consequence of Theorem 12. It is essentially shown in [1, Theorem 3].

Theorem 38 (Theorem 3 of [1])**.** *Let G be a graph with color classes X and $V \setminus X$, and let $M = \mathcal{M}_G * X$ be a corresponding binary matroid. Then $q_1(G)(y-1) = t_M(y,y)$.*

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